MORPHOLOGICAL PDES ON GRAPHS FOR ANALYZING UNORGANIZED DATA IN 3D AND HIGHER

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ABSTRACT

Mathematical morphology operators can be defined in terms of algebraic (discrete) sets or as partial differential equations (PDEs). In our previous works [1, 2], we have proposed a simple method to solve PDEs (Partial Differential Equations) on dataset using the framework PdEs (Partial difference Equations) on graphs. In this paper, we propose to apply morphological-based operators on unorganized dataset.

Index Terms— Generalized distance, Weighted graphs, Unorganized data, Partial difference equations, Mathematical morphology

1. INTRODUCTION

Mathematical morphology (MM) is a popular nonlinear approach for image processing that has found numerous applications including shape and texture analysis, biomedical image processing, document recognition or multiresolution techniques. MM offers a wide range of operators to address various image processing problems. These operators can be defined in terms of algebraic (discrete) sets or as partial differential equations (PDEs).

In our previous works [1, 2], we have proposed a simple method to solve PDEs (Partial Differential Equations) on dataset using the framework PdEs (Partial difference Equations) on graphs. We formulated mathematical morphology operators (dilation and erosion) which can be used to perform several morphological processes on weighted graphs, such as opening, closing, reconstruction and leveling.

In this paper, we adopt the PdE method and we focus on some PDEs-based continuous morphological operators on Euclidean domains : Dilation/Erosion, Eikonal Equation. One strong benefit of our approach is that it enables to process any information associated to raw dataset.

The paper is organized as follows. In section 2, we present Partial difference operators on graphs. After that, in section 3, we present Morphological operators on graphs. In section 4, we present the construction of a weighted graph from unorganized data and experiment morphological based processing on 3D point cloud and higher dimensional data such as databases. Last section concludes.

2. PARTIAL DIFFERENCE OPERATORS ON GRAPHS

In this section, we recall definitions and operators on graphs. This constitutes the basis of the framework of PdEs on a graph [3] that enables to transpose PDEs on graphs. All these definitions are borrowed from [4, 5, 1].

2.1. Notations and Preliminaries

A weighted graph $G = (V, E, w)$ consists of a finite set $V = \{v_1, \ldots, v_N\}$ of N vertices and a finite set $E \subset V \times V$ of weighted edges. We assume G to be undirected, with no self-loops and no multiple edges. Let (u, v) be the edge of E that connects two vertices u and v of V. Its weight, denoted by $w(u, v)$, represents the similarity between its vertices. Similarities are usually computed by using a positive symmetric function $w : V \times V \to \mathbb{R}^+$ satisfying $w(u, v) = 0$ if $(u, v) \notin E$. The notation $u \sim v$ is also used to denote two adjacent vertices. The degree of a vertex u is defined as $\delta_w(u) = \sum_{v\sim u} w(u, v)$. A function $f : V \to \mathbb{R}$ of $H(V)$ assigns a real value $f(u)$ to each vertex $u \in V$.

2.2. Difference Operators on Weighted Graphs

Let $G = (V, E, w)$ be a weighted graph, $f : V \to \mathbb{R}$ be a function of $H(V)$ and $w : V \times V \to \mathbb{R}^+$, a weight function defined as a similarity measure between two vertices.

The *directional derivative* (or *edge derivative* of f, at a vertex $u \in V$, along an edge $e = (u, v)$, is defined as :

$$\partial_v f(u) = \sqrt{w(u,v)}(f(v) - f(u)). \quad (1)$$

The external and internal *morphological directional partial derivative* operators are respectively defined as [5] :

$$\partial_v^+ f(u) = \big(\partial_v f(u)\big)^+, \quad (2)$$

$$\partial_v^- f(u) = \big(\partial_v f(u)\big)^-. \quad (3)$$

where $(x)^+ = \max(x,0)$ and $(x)^- = -\min(x,0)$.

Discrete upwind non-local weighted gradients are defined as :

$$(\nabla_w^{\pm} f)(u) = \left((\partial_v^{\pm} f)(u)\right)^T_{v\in V}, \tag{4}$$

with w in subscript corresponds to the weight function defined on graphs.

The $\mathcal{L}_p$ norms and the $\mathcal{L}_\infty$ of these gradients are defined by :

$$||(\nabla_w^{\pm} f)(u)||_p = \left[\sum_{v\sim u} w(u,v)^{\frac{p}{2}} \left[(f(v)-f(u))^{\pm}\right]^p\right]^{\frac{1}{p}}, \tag{5}$$

$$||(\nabla_w^{\pm} f)(u)||_\infty = \max_{v\sim u}\left(w(u,v)|(f(v)-f(u))^{\pm}|\right). \tag{6}$$

3. MORPHOLOGICAL OPERATORS ON GRAPHS

In this section based on discrete gradient on weighted graphs, we present a class of discrete equation, that mimic PDEs-based definition of erosion and dilation, Eikonal Equation.

3.1. Dilation and Erosion on graphs for filtering

The dilation and erosion of an initial function $f^0 : V \to \mathbb{R}$ is defined by [1] :

$$\begin{aligned}\frac{\partial f}{\partial t}(u,t) &= +||(\nabla_w^+ f)(u)||_p\\ \frac{\partial f}{\partial t}(u,t) &= -||(\nabla_w^- f)(u)||_p,\end{aligned} \tag{7}$$

for $1 \le p \le \infty$, with initial condition $f(u,0) = f^0(u)$. The discrete expression of internal and external gradient constitue direct spectral numerical scheme, with the usual notation $f^n(u) = f(u, n\Delta t)$, the generation iterative scheme for dilation and erosion can be defined as :

$$f^{n+1}(u) = f^n(u) \pm \Delta t||\nabla_w^{\pm} f^n(u)||_p \qquad \text{with } f^{(0)} = f^0(u) \tag{8}$$

with $\mathcal{L}_p$ norm the equation becomes :

$$f^{n+1}(u) = f^{(n)}(u) \pm \Delta t\left[\sum_{v\in V}(w(u,v))^{\frac{p}{2}}|\partial_v^{\pm} f^n(u)|^p\right]^{\frac{1}{p}} \tag{9}$$

$$f^{(0)} = f^0(u) \tag{10}$$

Note in this general form, we have an algebric equation, in which coefficients can be depend on data.

For $w = 1$ and for a grid graph, this scheme corresponds to the first order discretization of Osher and Sethian scheme.

Now with $\mathcal{L}_\infty$ norm this equation becomes :

$$f^{(n+1)}(u) = f^{(n)}(u) \pm \Delta t \max_{v\sim u}(\sqrt{w(u,v)}|\partial_v^{\pm} f^{(n)}(u)|) \tag{11}$$

with $\Delta t = 1$ and $w = 1$ (unweighted graph) the dilation and erosion become respectively :

$$f^{(n+1)} = \max_{v\sim u}(f^{(n)}(v), f^{(n)}(u)) \tag{12}$$

$$f^{(n+1)} = \min_{v\sim u}(f^{(n)}(v), f^{(n)}(u)) \tag{13}$$

In the special case where $\Delta t = 1$, the dilation PdE can be interpreted as an iterative non-local dilation (NLD) process, and as a non-local erosion (NLE) for the erosion PdE. These processes can be expressed as

$$f^{n+1}(u) = NLD(f^n)(u) = f(u) + ||(\nabla_w^+ f)(u)||_\infty, \tag{14}$$

for the dilation, and

$$f^{n+1}(u) = NLE(f^n)(u) = f(u) - ||(\nabla_w^- f)(u)||_\infty, \tag{15}$$

for the erosion.

This approach can be used to define other morphological operators based on erosion ϵ or dilation δ operators, such as openings $\gamma = (\delta\epsilon)$, closings $\phi = (\epsilon\delta)$, or morphological gradients $(\delta - \epsilon)$. For instance we propose a formulation of the non-local closing (NLC) operation that can be defined as :

$$\begin{aligned}\frac{\partial f}{\partial t}(u,t) = &- sign^+(t-s+1)||(\nabla_w^- f)(u)||_\infty\\ &+ sign^+(s-t)||(\nabla_w^+ f)(u)||_\infty,\end{aligned} \tag{16}$$

with $t \in [0, 2s[$, $s \in \mathbb{R}^+$ and

$$sign^+(x) = \begin{cases}1 \text{ if } x > 0,\\ 0 \text{ otherwise.}\end{cases} \tag{17}$$

This PdE has a time-dependent switching coefficient that makes it act as a dilation δ for $t \in [0, s[$ and an erosion ϵ for $t \in [s, 2s[$. This formulation is different from the classical PDEs one [6] and does not produce discontinuities at the switching time. This can be interpreted as the following non-local iterative process :

$$\begin{aligned}f^{(n+1)}(u) &= NLC(f^{(n)})(u)\\ &= sign^+(t-s+1)NLE(f^{(n)})(u)\\ &\quad + sign^+(s-t)NLD(f^{(n)})(u).\end{aligned} \tag{18}$$

Similarly, one can express the non-local opening NLO as an iterative process.

3.2. Eikonal Equation on graphs

Let $G(V, E, w)$ be a weighted graph. A front evolving on G is defined as a subset $\Omega_0 \subset V$, and is implicitly represented at initial time by a level set function $\phi_0 = \mathcal{U}_0 = \chi_{\Omega_0} - \chi\Omega_0^c$, where $\chi : V \to \{0,1\}$ is the indicator function and Ω_0^c is the complement of Ω_0. In other words ϕ_0 equals 1 in Ω_0 and -1

on its complementary.
The front propagation can be described by :

$$\begin{cases} \dfrac{\partial \phi}{\partial t}(u) & = \mathcal{F}(u)||(\nabla_w \phi)(u)||_\infty \\ \phi_0(u) & = \mathcal{U}_0 \end{cases} \tag{19}$$

with $\mathcal{F} \in \mathcal{H}(V)$, and $w : V \times V \to \mathbb{R}^+$ is the weight function.

In the case where $\mathcal{F}$ is defined non-negative on the whole domain Ω, the relation between the level set formulation (19) and the well-known Eikonal Equation $(\mathcal{F}(x)||\nabla T(x)|| = 1)$ stems from the following change of variable : $\phi(x, y) = t - T(x)$ (where $T(x)$ is the arrival time of the curve at a point x).
Using previous definitions of morphological evolution equations, one can formulate the same relation and obtain a PdEs-based version of the Eikonal Equation, defined on weighted graphs of arbitrary topology [7]. Because $\mathcal{F}$ is defined non-negative, the evolution process described by Eq. (19) can be seen as a dilation process and the evolution equation rewritten as :

$$\frac{\partial \phi}{\partial t}(u) = \mathcal{F}||(\nabla_w^+ \phi)(u)||. \tag{20}$$

With a similar change of variable $\phi(u) = t - T(u)$, we have

$$\frac{\partial \phi}{\partial t}(u) = \mathcal{F}||(\nabla_w^+ (t - T)(u)|| = \mathcal{F}||(\nabla_w^- T)(u)|| = 1. \tag{21}$$

Finally, with $P = 1/\mathcal{F}$ we obtain a discrete adaptation of the Eikonal Equation on weighted graph, which descibes a morphological erosion process, and defined as :

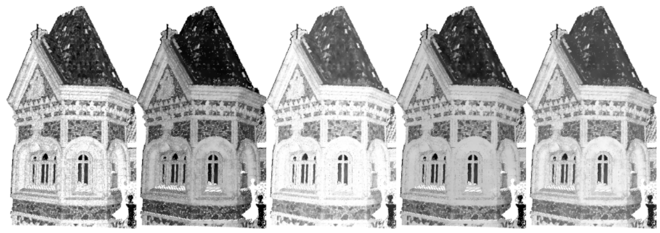
$$\begin{cases} ||(\nabla_w^- f)(u)||_p = P(u) & \forall u \in V \\ f(u) = 0 & \forall u \in V_0 \end{cases} \tag{22}$$

4. GRAPH CONSTRUCTION & EXPERIMENTS

In this section, we explain how a weighted graph can be associated with unorganized data. Next, we show some examples of restoration and segmentation with the proposed morphological filters on graphs constructed from arbitrary data.

4.1. From unorganized data to graph construction

First step consists in defining the sets V and E from a given dataset. Let us consider a dataset P as a set of data points $\{\vec{p}_1, \ldots, \vec{p}_n\} \in \mathbb{R}^n$. There are many ways of associating a graph, that encodes proximity between points, to such a data set. To each data point we first associate a vertex of a proximity graph G to define a set of vertices $V = \{v_1, v_2, \ldots, v_n\}$. Then, determining the edge set E of the proximity graph G requires defining the neighbors of each vertex v_i according to its embedding $\vec{p}_i$ using the Euclidean distance. We consider the k Nearest Neighbors Graph (k-NNG) : $v_j \sim v_i$ if the distance between $\vec{p}_i$ and $\vec{p}_j$ is among the k-th smallest distances from $\vec{p}_i$ to all the other data points. To conclude, the first step consists in associating a k-NNG to the dataset. The value of k will be denoted k_G for the graph G associated with the dataset. To speed up the knn algorithm, a kD-tree can also be used [8].



(a) Original (b) NLE (c) NLD (d) NLO (e) NLC

Fig. 1: Morphological filtering on a colored point cloud after 10 iterations with $k_G = 15$ with operators defined in sub-section 3.1. Gracefully given by `www.cloudcasterlite.com`.

Once the graph has been created, it has to be weighted. If one does not want to take care of the vertices similarities, the weight function w can be set to $w = 1$. A better one can be obtained using patches [9]. For images, a patch $\vec{\mathcal{P}}(v_i)$ centered at a vertex $v_i \in V$ is a vector of values (e.g., coordinates, intensities, ...) defined by $\vec{\mathcal{P}}(v_i) = \left(f^0(v_j) : v_j \in B(v_i, n)\right)^T$ where $B(v_i, n)$ is a square of size n^2 centered at v_i. Using patches, $w : V \times V \to \mathbb{R}$ is defined by :

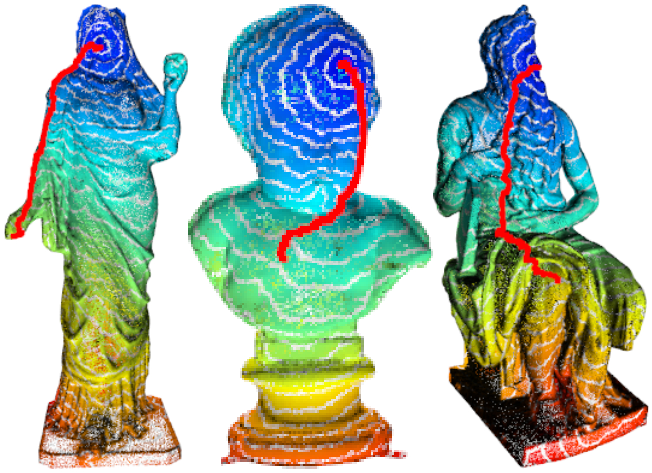
$$w(v_i, v_j) = exp\left(-\frac{||\vec{\mathcal{P}}(v_i) - \vec{\mathcal{P}}(v_j)||_2^2}{\sigma^2}\right). \tag{23}$$

Extending the notion of a patch to three-dimensional point cloud data is not an easy task. We have proposed a novel definition of patches that can be used for any graph representation associated to meshes or 3D point clouds, see [10] for more details.

4.2. Experiments

Filtering of colored Point Clouds. Let P be a point cloud, that associates an intensity to each point $p \in P$. From the latter point cloud, a weighted graph $G = (V, E, w)$ is first created using the method presented in sub-section 4.1 (i.e., a k_G nearest neighbor graph). Then this graph is filtered using some morphological operators, as explained in sub-section 3.1. Figure 1 shows some morphological processings of a tower of the bishop point cloud.

Shortest path computing on Point Clouds. We compute the generalized distance on several point clouds by solving Eikonal Equation, see Figure 2. We built the graph as a k-nn with $k = 5$, in the coordinate space of the point cloud. As the spatial discretization step is regular enough, we used a constant weight function ($w(u, v) = 1$). The superimposed



(a) (b) (c)

Fig. 2: Compute of geodesic distance on several 3D point clouds with the Eikonal Equation on graph. Bottom line show evolution of distance with minimal path linking two points of the point cloud. Original data graciously provided by `http://sketchfab.com/` under Creative Commons licence.

red line on the figure is the shortest path between the source point (the point from which the distance is computed) and an other point in the point cloud. This path was obviously computed using the computed distance function.

Clustering on unorganized database. The Information Retrieval (IR) task consists in matching objects stored in a database. The United States Postal Service (USPS) handwritten digits database contains grayscale handwritten digit images scanned from digit 0 to 9 where each image is of size 16×16 pixels. In our experiment, we show the 50 first samples founded classified according distances computed on graphs (see Figure 3).

5. CONCLUSION

In this paper, we adopted the PdE method and we focused on some PDEs-based continuous morphological operators on Euclidean domains : Dilation/Erosion, Eikonal Equation. We briefly presented the construction of a weighted graph from unorganized data and show several examples of processing such as the filtering, the computing of the shortest path or the clustering on unorganized data. The presented works carried out within the Graph Signal Processing project.

6. REFERENCES

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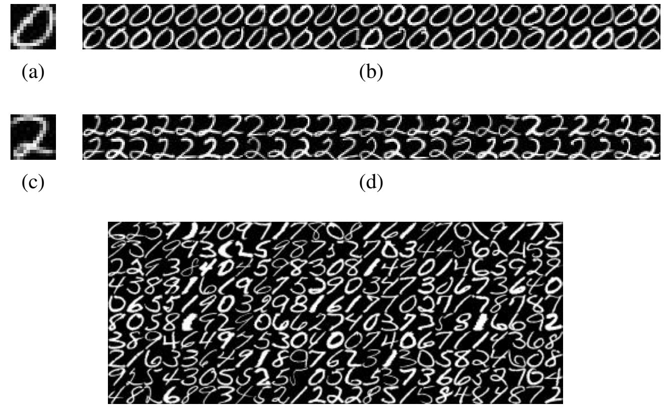

(e)

Fig. 3: USPS image retrieval based on distances computed on graphs (a) and (c) : user input query . (b) and (d) : 50 first obtained results for the corresponding query. (e) : sample of the initial data set.

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