Nonlocal difference operators on Graphs for interpolation on Point Clouds

François Lozes and Abderrahim Elmoataz

Université de Caen Basse-Normandie and the ENSICAEN in the GREYC UMR CNRS 6072 Laboratory, Image Team, 6 Boulevard Maréchal Juin, F-14050 CAEN cedex FRANCE.
{abderrahim.elmoataz-billah,francois.lozes}@unicaen.fr

Abstract. In this paper we introduce a new general class of partial difference operators on graphs, which interpolate between the nonlocal ∞-Laplacian, the Laplacian, and a family of discrete gradient operators. In this context we investigate an associated Dirichlet problem for this general class of operators and prove the existence and uniqueness of respective solutions. We propose to use this class of operators as general framework to solve many interpolation problems in a unified manner as arising, e.g., in image and point cloud processing.[1]

1 Introduction

Partial differential equations (PDEs) involving the p-Laplace and ∞-Laplace operators still generate a lot of interest both in the setting of Euclidean domains as well as on discrete graphs. These operators in their different forms, i.e., continuous, discrete, local, and nonlocal, are at the interface of many scientific fields as they are used to model many interesting phenomena, e.g., in mathematics, physics, engineering, biology, and economy. Some closely related applications can be found in image processing, computer vision, machine learning, and game theory, see e.g., [4, 7, 1] and references therein.

In this paper, we introduce and study a novel adaptive general class of Partial difference Equations (PdEs) on weighted graphs. These equations are based on finite difference operators which adaptively interpolate between two discrete upwind gradients and the p-Laplacian operator on graphs. The advantage of the involved family of operators is its adaptivity with respect to potential applications, i.e., to handle many local and nonlocal interpolation problems in image and data processing within the same unified framework, e.g., inpainting, colorization, and semi-supervised clustering.

Furthermore, in order to solve the associated Dirichlet problem in the setting of discrete graphs we propose an algorithm which can be used to unify interpolation tasks on both conventional images and point cloud data.

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The rest of this work is organized as follows. In Section 2 we provide basic definitions and notations, which are used throughout this work. Furthermore, we recall our previous works on PdEs on graphs and the p-Laplacian on graphs. In Section 3 we derive a novel partial difference operator which interpolates between the graph p-Laplacian, the graph ∞-Laplacian, and discrete gradient operators. Then, we study an associated Dirichlet problem and prove the existence and uniqueness of respective solutions. Section 4 presents several applications and interpolation problems on real world images and point clouds. Finally, a short discussion in Section 5 concludes this paper.

2 Partial difference operators on graphs

In this section we introduce the basic notations used throughout this paper. Additionally, we recall various definitions of difference operators on weighted graphs from previous works in order to define in this context derivatives, the p-Laplace operator, and some morphological operators on graphs. More details on these operators can be found in [11, 5, 18].

2.1 Notations and Preliminaries

A weighted graph $G = (V, E, w)$ consists of a finite set $V = \{v_1, \ldots, v_N\}$ of N vertices and a finite set $E \subset V \times V$ of weighted edges. We assume G to be undirected, with no self-loops and no multiple edges. Let (u, v) be the edge of E that connects two vertices u and v of V. Its weight, denoted by $w(u, v)$, represents the similarity between its vertices. Similarities are usually computed by using a positive symmetric function $w: V \times V \to \mathbb{R}^+$ satisfying $w(u, v) = 0$ if $(u, v) \notin E$. The notation $u \sim v$ is also used to denote two adjacent vertices. The degree of a vertex u is defined as $\delta_w(u) = \sum_{v \sim u} w(u, v)$. A function $f: V \to \mathbb{R}$ of $H(V)$ assigns a real value $f(u)$ to each vertex $u \in V$.

The two *upwind gradient norm operators* $\|\nabla_w^{\pm} f\|_{\infty}$: $\mathcal{H}(V) \to \mathcal{H}(V)$ for a function $f \in \mathcal{H}(V)$ can be defined as :

$$\|(\nabla_w^{\pm} f)(u)\|_{\infty} = \max_{v \sim u} \Big(\sqrt{w(u, v)} \big(f(v) - f(u)\big)^{\pm} \Big), \tag{1}$$

where $(x)^+ = \max(x, 0)$ and $(x)^- = -\min(x, 0)$.

2.2 Morphological Nonlocal Dilation, Erosion and Mean on graphs

These gradients were also used to approximate certain continuous Hamilton-Jacobi equations on a discrete domain [18, 10]. For example, given two functions $f, \mu : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, then any continuous equation of the form:

$$\frac{\partial f(x, t)}{\partial t} = \mu(x) \|\nabla f(x, t)\|_p, \tag{2}$$

can be numerically approximated in a discrete setting as :

$$\frac{\partial f(u,t)}{\partial t}{=}\mu^+(u)\|\big(\nabla_w^+ f\big)(u,t)\|_p - \mu^-(u)\|\big(\nabla_w^- f\big)(u,t)\|_p, \tag{3}$$

where $\mu^+(u) = \max(\mu(u), 0)$ and $\mu(u)^- = -\min(\mu(u), 0)$.

In particular, if $\mu \equiv 1$, $p = \infty$, and if we employ a forward Euler time discretization with $\Delta t = 1$ (for stability we have $\Delta t \leq 1$), this equation can be rewritten as :

$$f^{k+1}(u) = f^k(u) + \|\big(\nabla_w^+ f^k\big)(u)\|_\infty, \tag{4}$$

with $f^k(u) = f(u, k\Delta t)$. This can be interpreted as a single iteration of the following *nonlocal dilation* type operator :

$$f^{k+1}(u) = NLD\big(f^k\big)(u), \tag{5}$$

where $NLD : \mathcal{H}(V) \to \mathcal{H}(V)$ is defined as :

$$NLD\big(f\big)(u) = f(u) + \max_{u\sim v}\Big(\sqrt{w(u,v)}\big(f(v) - f(u)\big)^+\Big). \tag{6}$$

Similarly, for the case $\mu \equiv -1$ and $p = \infty$ we have $NLE : \mathcal{H}(V) \to \mathcal{H}(V)$ defined as :

$$NLE\big(f\big)(u) = f(u) - \max_{u\sim v}\Big(\sqrt{w(u,v)}\big(f(v) - f(u)\big)^-\Big). \tag{7}$$

Likewise, for the case of the continuous Laplacian, the discretization leads to the operator $NLM : \mathcal{H}(V) \to \mathcal{H}(V)$, which is the well-known nonlocal mean filter [6], defined as

$$NLM(f)(u) = \frac{\sum_{v\sim u} w(u,v)f(v)}{\delta_w(u)}. \tag{8}$$

3 A new family of graph adaptive operators

In this section we propose a new family of discrete operators on weighted graphs which corresponds to a graph operators with gradients terms and we investigated an associated Dirichlet problem.

3.1 Definition

Based on the discussed PdE framework on graphs in Section 2, we are now able to propose a novel family of operators denoted by $\Delta_{\alpha,\beta,\gamma}\colon \mathcal{H}(V) \to \mathcal{H}(V)$ for a function $f \in \mathcal{H}(V)$ by:

$$\Delta_{\alpha,\beta,\gamma} f(u) \;=\; \alpha(u)\|\nabla_w^+ f(u)\|_\infty - \beta(u)\|\nabla_w^- f(u)\|_\infty + \gamma(u)\Delta_{w,2} f(u), \tag{9}$$

with $u \in V$, $\alpha(u), \beta(u), \gamma(u) : V \to \mathbb{R}$ and $\alpha(u) + \beta(u) + \gamma(u) = 1$. By a simple factorization of the ∞-Laplacian this new family of operators can be rewritten as :

$$\begin{aligned} \Delta_{\alpha,\beta,\gamma} f = & \, 2\min(\alpha(u), \beta(u)) \Delta_{w,\infty} f(u) + (\alpha(u) - \beta(u))^{+} \|\nabla_w^{+} f(u)\|_{\infty} \\ & - (\alpha(u) - \beta(u))^{-} \|\nabla_w^{-} f(u)\|_{\infty} + \gamma \Delta_{w,2} f(u). \end{aligned} \tag{10}$$

With $\alpha(u), \beta(u), \gamma(u)$ constants, we retrieve formulation presented in [12]. We propose to defined $\alpha(u), \beta(u), \gamma(u)$ as :

$$\alpha(u) = \frac{\sum_{f(v)-f(u)>\epsilon} w(u,v)}{\delta_w(u)} \qquad \beta(u) = \frac{\sum_{f(v)-f(u)<\epsilon} w(u,v)}{\delta_w(u)}, \tag{11}$$

and $\gamma(u) = 1 - \alpha(u) - \beta(u)$. Note that this family of operators is directly related to the nonlocal average operator : $\Delta_{\alpha,\beta,\gamma} f = NLA(f) - f$, for which we refer to the operator $NLA : \mathcal{H}(V) \to \mathcal{H}(V)$ as 'Nonlocal Average' with

$$NLA(f)(u) = \alpha(u) NLD(f)(u) + \beta(u) NLE(f)(u) + \gamma(u) NLM(f)(u), \tag{12}$$

and the operators NLD, NLE, and NLM as introduced in Section 2.

3.2 Dirichlet problem

In the following we focus on a PdE related to the proposed family of graph operators with gradient terms. In particular, we investigate an associated Dirichlet problem. Let $G = (V, E, w)$ be an undirected, weighted, and connected graph, $A \subset V$ a subset of vertices, the boundary of A defined as $\partial A = V \backslash A$ and $g : \partial A \to \mathbb{R}$. We consider the PdE as :

$$\begin{cases} \big(\Delta_{\alpha,\beta,\gamma} f\big)(u) = 0, & u \in A, \\ f(u) = g(u), & u \in \partial A, \end{cases} \tag{13}$$

for the general case $\gamma \neq 0$. We could demonstrate as in [12] that the problem (13) has a unique solution.

4 Unified interpolation for inverse problems on images and point clouds

Many tasks in computer vision and image processing can be formulated as interpolation problems. Image and video colorization [15], inpainting [2, 17], and semi-supervised segmentation [13, 19] are examples of these interpolation problems. In general, interpolation consists of estimating appropriate values in regions of missing data while staying coherent with respect to the given data. Until today many methods have been developed and proposed for image interpolation [13, 6, 11, 18]. Among them, a significant amount of methods is based on local or nonlocal PDEs or variational methods.

(a) (b) (c) (d) (e)

Fig. 1: Colorization of points clouds. (a) uncolored dwarf, (b) half-colored dwarf, (c) full colored dwarf, (d) original point cloud, (e) colorized point cloud.

In this work we propose to use the in Section 3 introduced family of graph operators as a unified framework. Among other tasks, this framework can be used to solve semi-supervised segmentation or clustering, image inpainting, as well as colorization of point clouds. To perform this task we propose to solve the discussed Dirichlet problem from (13), for which $A \subset V$ is the subset of vertices associated to the missing information. Note that the initial value function g is application-dependent and will be defined for each application in the sequel.

To solve (13) we make use of the following associated evolution equation problem:

$$\begin{cases} \frac{\partial}{\partial t} f(u,t) = \Delta_{\alpha,\beta,\gamma} f(u,t), & u \in A, \\ f(u,t) = g(u), & u \in \partial A, \\ f(u,t=0) = f_0(u), & u \in A, \end{cases} \tag{14}$$

for which f_0 is an initial function that is also application-dependent. To solve (14) iteratively we use an explicit forward Euler time discretization. Using $\Delta_{\alpha,\beta,\gamma} = NLA(f) - f$ and setting $\Delta t = 1$, we get the following nonlocal average filter:

$$\begin{cases} f^{n+1}(u) = NLA(f^n)(u) & u \in A, \\ f^{n+1}(u) = g(u), & u \in \partial A, \\ f^0(u) = f_0(u), & u \in A. \end{cases} \tag{15}$$

Graph construction : The first step in the graph construction, consists in defining the sets V and E from a given dataset. Let us consider a dataset P as a set of data points $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\} \in \mathbb{R}^n$. To each data point we first associate a vertex of a proximity graph G to define a set of vertices V. Then, we determine the edge set E from the neighbors of each vertex v_i. We consider the k Nearest Neighbors Graph (k-NNG): $v_j \sim v_i$ if the distance between $\boldsymbol{p}_i$ and $\boldsymbol{p}_j$ is among the k-th smallest distances from $\boldsymbol{p}_i$ to all the other data points. To speed up the k-NN algorithm, a kD-tree can also be used [3].

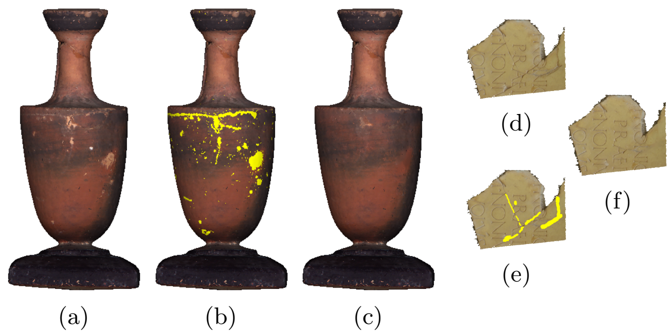

Fig. 2: Restoration of antique objects. (a) Original vasis, (b) Vasis to inpaint, (c) Restored vasis, (d) Original wall, (e) Wall to inpaint, (f) Restored wall.

Once the graph has been created, it has to be weighted. If one does not want to take care of the vertices similarities, the weight function w can be set to $w = 1$. A better one can be obtained using patches [6]. For images, a patch $\boldsymbol{\mathcal{P}}(v_i)$ centered at a vertex $v_i \in V$ is a vector of values (e.g., coordinates, intensities) defined by $\boldsymbol{\mathcal{P}}(v_i) = \big(f^0(v_j) : v_j \in B(v_i, n)\big)^T$ where $B(v_i, n)$ is a square of size n^2 centered at v_i. Using patches, $w : V \times V \to \mathbb{R}$ is defined by: $w(v_i, v_j) = exp\left(-\frac{\|\boldsymbol{\mathcal{P}}(v_i)-\boldsymbol{\mathcal{P}}(v_j)\|_2^2}{\sigma^2}\right)$. We have proposed a novel definition of patches to three-dimensional point cloud that can be used for any graph representation associated to meshes or 3D point clouds, see [16] for more details.

3D colorization : Image colorization is the process of adding colors to monochromatic images. To colorize monochrome images the luminance channel is used to determine pixels similarities which enable color diffusion from scribbles. In the case of 3D data however, the intensity channel is missing and similarities between points have to be determined in a different way. To the best of our knowledge Leifman and Tal [14] are the only researchers which have proposed a method for *mesh colorization* up to now. The colorization is then performed by solving a constrained quadratic optimization problem (as in [15]). Let $f^0 : V \to \mathbb{R}^3$ be a function that assigns RGB colors to vertices. Let $A \subset V$ be the subset of vertices with unknown colors and ∂A the subset of vertices for which $g : \partial A \to \mathbb{R}^3$ gives the user-specified color scribbles. Then, we are able to use the iteration scheme (15) to perform 3D colorization of point cloud data. Figure 1 shows results of the method to colorize several 3D point clouds.

Nonlocal Inpainting : Digital inpainting can be simply formulated as reconstructing a damaged or incomplete image by filling the missing information in certain regions. In recent years many methods have been developed for interpolating geometry [8], texture [9], or both geometry and texture [2]. Among the proposed interpolation methods a significant number of algorithms are based on PDEs or variational methods, see e.g., [2] and references therein. Recent works tend to unify local and nonlocal interpolation approaches [13]. With respect to

(a) (b) (c)

Fig. 3: Illustration of segmentation on a colored 3D point cloud. See text for more details. (a) Original, (b) Label, (c) Segmentation result.

(13) we propose to formulate the inpainting problem as follows: A is the set of pixels with missing information, $g : \partial A \rightarrow \mathbb{R}^c$, represents the known information (for which c is the number of color channels of the image), and $f : A \rightarrow \mathbb{R}^c$ represents the image to be reconstructed. Using this notation we are able to use the iteration scheme (15) to perform nonlocal inpainting. We illustrate this approach in Figure 2 to inpaint the texture reconstruction on colored 3D point cloud data.

Semi-supervised segmentation : We propose to consider the semi-supervised segmentation task as an interpolation problem, for which the function to be interpolated is the label function specifying the partition. Considering a partition into two classes A and B, with $N = 2$ the number of classes to segment. A multi-phase segmentation can be performed by applying the iteration scheme (13) N times and considering the label A as a class and B as the other classes. In this case, the label function L, associating a class to each vertex, defined as $L : V \rightarrow \{C_i\}_{i=1,\dots,N}$ with $\{C_i\}$ the set of class labels is computed as : $L(u) = C_i | f_i(u) = \max\limits_{j=1,\dots,N} f_j(u)$. Figure 3 shows exemplary results of the method to segment a 3D colored point cloud. The graph is built in a similar way as in the subsection 4.

5 Conclusion

In this paper we have introduced a novel family of graph operators with gradient terms. These partial difference operators interpolate between nonlocal ∞-Laplacian, nonlocal Laplacian, and gradient terms on graphs. We considered an associated Dirichlet problem for this class of operators and have proven the existence and uniqueness of respective solutions. Finally, we have demonstrated the applicability of these operators in terms of a unified framework to solve many inverse problems in image processing, 3D point cloud processing.

Bibliography

[1] Andreu, F., Mazón, J., Rossi, J., Toledo, J.: A nonlocal p-laplacian evolution equation with neumann boundary conditions. Journal de mathématiques pures et appliquées 90(2), 201–227 (2008)

[2] Arias, P., Facciolo, G., Caselles, V., Sapiro, G.: A variational framework for exemplar-based image inpainting. IJCV 93(3), 319–347 (2011)

[3] Arya, S., Mount, D.M., Netanyahu, N.S., Silverman, R., Wu, A.Y.: An optimal algorithm for approximate nearest neighbor searching fixed dimensions. J. ACM 45(6), 891–923 (Nov 1998)

[4] Bertozzi, A.L., Flenner, A.: Diffuse interface models on graphs for classification of high dimensional data. Multiscale Modeling & Simulation 10(3), 1090–1118 (2012)

[5] Bougleux, S., Elmoataz, A., Melkemi, M.: Local and nonlocal discrete regularization on weighted graphs for image and mesh processing. International Journal of Computer Vision 84(2), 220–236 (2009)

[6] Buades, A., Coll, B., Morel, J.M.: Nonlocal image and movie denoising. IJCV 76(2), 123–139 (2008)

[7] Bühler, T., Hein, M.: Spectral clustering based on the graph p-laplacian. In: Proceedings of the 26th Annual ICML. pp. 81–88. ACM (2009)

[8] Chan, T.F., Kang, S.H., Shen, J.: Euler's elastica and curvature-based inpainting. SIAM Journal on Applied Mathematics pp. 564–592 (2002)

[9] Criminisi, A., Pérez, P., Toyama, K.: Region filling and object removal by exemplar-based image inpainting. Image Processing, IEEE Transactions on 13(9), 1200–1212 (2004)

[10] Desquesnes, X., Elmoataz, A., Lézoray, O.: Eikonal equation adaptation on weighted graphs: fast geometric diffusion process for local and non-local image and data processing. JMIV 46(2), 238–257 (2013)

[11] Elmoataz, A., Lezoray, O., Bougleux, S.: Nonlocal discrete regularization on weighted graphs: a framework for image and manifold processing. Image Processing, IEEE Transactions on 17(7), 1047–1060 (2008)

[12] Elmoataz, A., Lozes, F., Toutain, M.: Nonlocal pdes on graphs: From tug-of-war games to unified interpolation on images and point clouds. JMIV pp. 1–21 (2016)

[13] Gilboa, G., Osher, S.: Nonlocal linear image regularization and supervised segmentation. Multiscale Modeling & Simulation 6(2), 595–630 (2007)

[14] Leifman, G., Tal, A.: Mesh colorization. Comput. Graph. Forum 31(2), 421–430 (2012)

[15] Levin, A., Lischinski, D., Weiss, Y.: Colorization using optimization. ACM Trans. Graph. 23(3), 689–694 (Aug 2004)

[16] Lozes, F., Elmoataz, A., Lezoray, O.: Partial difference operators on weighted graphs for image processing on surfaces and point clouds. Image Processing, IEEE Transactions on 23(9), 3896–3909 (Sept 2014)

[17] Schönlieb, C.b., Bertozzi, A.: Unconditionally stable schemes for higher order inpainting. Communications in Mathematical Sciences 9(2), 413–457 (2011)

[18] Ta, V.T., Elmoataz, A., Lézoray, O.: Nonlocal pdes-based morphology on weighted graphs for image and data processing. Image Processing, IEEE Transactions on 20(6), 1504–1516 (2011)

[19] Zhou, D., Schölkopf, B.: Discrete regularization. In: Semi-Supervised Learning, pp. 221–232. MIT Press (2006)